

# $\mathcal{PT}$ -Symmetric Quantum Electrodynamics— $\mathcal{PT}$ QED

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**Abstract.** The construction of  $\mathcal{PT}$ -symmetric quantum electrodynamics is reviewed. In particular, the massless version of the theory in  $1 + 1$  dimensions (the Schwinger model) is solved. Difficulties with unitarity of the  $S$ -matrix are discussed.

## 1. $\mathcal{PT}$ QED

Quantum electrodynamics (QED) is by far the most successful physical theory ever devised [1]. However, although its reach includes all of atomic physics, there are a myriad of phenomena that we do not understand. Thus it is essential to explore alternative theories, in the hope that we may be able to describe aspects of the world that are as yet not under our understanding.

One very promising new approach to quantum theories are those included under the rubric of non-Hermitian theories, in particular theories in which invariance under the combined operation of space and time reflection  $\mathcal{PT}$  replaces mathematical Dirac Hermiticity in order to guarantee unitarity of the theory. (For a recent review, see [2].) Little work, however, has been done on applying this idea to quantum field theory. This paper represents our continuing effort to develop a  $\mathcal{PT}$ -symmetric version of quantum electrodynamics, in the hope that a consistent theory, with a unitary  $S$ -matrix, results that may eventually find physical applications in nature.

### 1.1. Transformation properties

At the first International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics [3] a  $\mathcal{PT}$ -symmetric version of quantum electrodynamics was proposed. A non-Hermitian but  $\mathcal{PT}$ -symmetric electrodynamics is based on the assumption of novel transformation properties of the electromagnetic fields under parity  $\mathcal{P}$  transformations, that is,

$$\mathcal{P} : \quad \mathbf{E} \rightarrow \mathbf{E}, \quad \mathbf{B} \rightarrow -\mathbf{B}, \quad \mathbf{A} \rightarrow \mathbf{A}, \quad A^0 \rightarrow -A^0, \quad (1)$$

just the statement that the four-vector potential is assumed to transform as an axial vector. Under time reversal  $\mathcal{T}$ , the transformations are assumed to be conventional,

$$\mathcal{T}: \quad \mathbf{E} \rightarrow \mathbf{E}, \quad \mathbf{B} \rightarrow -\mathbf{B}, \quad \mathbf{A} \rightarrow -\mathbf{A}, \quad A^0 \rightarrow A^0. \quad (2)$$

Fermion fields are also assumed to transform conventionally. We use the metric  $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ .

### 1.2. Lagrangian and Hamiltonian

The Lagrangian of the theory then possesses an imaginary coupling constant in order that it be invariant under the product of these two symmetries:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{i}\partial_\mu\psi - \frac{m}{2}\psi\gamma^0\psi + \frac{i}{2}\psi\gamma^0\gamma^\mu eq\psi A_\mu. \quad (3)$$

Here, because we are discarding Hermiticity as a physical requirement, it is most appropriate to use a “real” field formulation, where correspondingly the (antisymmetric, imaginary) charge matrix  $q = \sigma_2$  appears. Furthermore,  $\gamma^0\gamma^\mu$  is symmetric and  $\gamma^0$  is antisymmetric. In the radiation (Coulomb) gauge  $\nabla \cdot \mathbf{A} = 0$ , the dynamical variables are  $\mathbf{A}$  and  $\psi$ , and the canonical momenta are  $\pi_{\mathbf{A}} = -\mathbf{E}^T$ ,  $\pi_\psi = \frac{i}{2}\psi$ , where  $T$  denotes the transverse part, and so the relation between the Hamiltonian and Lagrangian densities are

$$\mathcal{H} = E^2 + \mathbf{E} \cdot \nabla A^0 + \frac{i}{2}\psi\dot{\psi} - \mathcal{L}. \quad (4)$$

Then, if integrate by parts and use  $\nabla \cdot \mathbf{E} = ij^0$ , we find that the corresponding Hamiltonian is

$$H = \int (d\mathbf{r}) \left\{ \frac{1}{2}(E^2 + B^2) + \frac{1}{2}\psi \left[ \gamma^0\gamma^k \left( \frac{1}{i}\nabla_k - ieqA_k \right) + m\gamma^0 \right] \psi \right\}. \quad (5)$$

We can also obtain this same Hamiltonian from the stress tensor

$$t^{\mu\nu} = -\frac{i}{4}\psi\gamma^0(\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu)\psi + F^{\mu\lambda}F^\nu{}_\lambda - \frac{i}{2}(j^\mu A^\nu + j^\nu A^\mu) + g^{\mu\nu}\mathcal{L}. \quad (6)$$

### 1.3. Current density

The electric current appearing in both the Lagrangian and Hamiltonian densities,

$$j^\mu = \frac{1}{2}\psi\gamma^0\gamma^\mu eq\psi, \quad (7)$$

transforms conventionally under both  $\mathcal{P}$  and  $\mathcal{T}$ :

$$\mathcal{P}j^\mu(\mathbf{x}, t)\mathcal{P} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(-\mathbf{x}, t), \quad (8a)$$

$$\mathcal{T}j^\mu(\mathbf{x}, t)\mathcal{T} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(\mathbf{x}, -t). \quad (8b)$$

This just reflects the normal transformation properties of the fermion fields.

#### 1.4. Equal-time commutation relations

We are working in the Coulomb gauge,  $\nabla \cdot \mathbf{A} = 0$ , so the nonzero canonical equal-time commutation relations are (the fermion index  $a$  includes both the Dirac and charge indices)

$$\{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \delta_{ab} \delta(\mathbf{x} - \mathbf{y}), \quad (9a)$$

$$[A_i^T(\mathbf{x}), E_j^T(\mathbf{y})] = -i \left[ \delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right] \delta(\mathbf{x} - \mathbf{y}). \quad (9b)$$

$$\nabla \cdot \mathbf{A}^T = \nabla \cdot \mathbf{E}^T = 0. \quad (10)$$

#### 1.5. The $\mathcal{C}$ operator

As for quantum mechanical systems, and for scalar quantum field theory, we seek a  $\mathcal{C}$  operator in the form

$$\mathcal{C} = e^{\mathcal{Q}} \mathcal{P}, \quad (11)$$

where  $\mathcal{P}$  is the parity operator.  $\mathcal{C}$  must satisfy the properties

$$\mathcal{C}^2 = 1, \quad (12a)$$

$$[\mathcal{C}, \mathcal{PT}] = 0, \quad (12b)$$

$$[\mathcal{C}, H] = 0. \quad (12c)$$

From the first two equations we infer

$$\mathcal{Q} = -\mathcal{P} \mathcal{Q} \mathcal{P}, \quad (13)$$

and because  $\mathcal{PT} = \mathcal{T} \mathcal{P}$ ,

$$\mathcal{Q} = -\mathcal{T} \mathcal{Q} \mathcal{T}. \quad (14)$$

The third equation (12c) allows us to determine  $\mathcal{Q}$  perturbatively. If we separate the interaction part of the Hamiltonian from the free part,

$$H = H_0 + eH_1, \quad (15)$$

and assume a perturbative expansion of  $\mathcal{Q}$ :

$$\mathcal{Q} = e\mathcal{Q}_1 + e^2\mathcal{Q}_2 + \dots, \quad (16)$$

the first contribution to the  $\mathcal{Q}$  operator is determined by

$$[\mathcal{Q}_1, H_0] = 2H_1. \quad (17)$$

The second correction commutes with the Hamiltonian,

$$[\mathcal{Q}_2, H_0] = 0. \quad (18)$$

Thus we may take

$$\mathcal{Q} = e\mathcal{Q}_1 + e^3\mathcal{Q}_3 + \dots, \quad (19)$$

which illustrates a virtue of the exponential form. The  $O(e)$  term was explicitly computed for four-dimensional QED in 2005 [4].

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However, the above perturbative construction of  $\mathcal{C}$  fails for 2-dimensional  $\mathcal{PT}$ -symmetric QED. We are here discussing the Schwinger model [5, 6, 7, 8]. In two dimensions, the only nonzero component of the field strength tensor is  $F^{01} = E$ , and the Hamiltonian of the system is

$$H = \int dx \left[ \frac{1}{2} E^2 - i j_1 A_1 - \frac{i}{2} \psi \gamma^0 \gamma^1 \partial_1 \psi + \frac{m}{2} \psi \gamma^0 \psi \right]. \quad (20)$$

As before, we choose the radiation gauge because it is most physical:

$$\nabla \cdot \mathbf{A} = \partial_1 A_1 = 0, \quad (21)$$

and then the Maxwell equation becomes

$$\partial_1 E_1 = -\partial_1^2 A^0 = i j^0, \quad (22)$$

which implies the following construction for the scalar potential

$$A^0(x) = -\frac{i}{2} \int_{-\infty}^{\infty} dy |x - y| j^0(y). \quad (23)$$

### 2.1. Construction of $E$

Without loss of generality, we can disregard  $A_1$ , and then the electric field is

$$E(x) = \frac{i}{2} \int_{-\infty}^{\infty} dy \epsilon(x - y) j^0(y), \quad (24)$$

with

$$\epsilon(x - y) = \begin{cases} 1, & x > y, \\ 0, & x = y, \\ -1, & x < y. \end{cases} \quad (25)$$

Thus the electric field part of the Hamiltonian is

$$\begin{aligned} \int dx \frac{1}{2} E^2 &= -\frac{1}{8} \int dx dy dz \epsilon(x - y) \epsilon(x - z) j^0(y) j^0(z) \\ &= -\frac{1}{8} L Q^2 + \frac{1}{4} \int dy dz j^0(y) |y - z| j^0(z), \end{aligned} \quad (26)$$

where  $L$  is the infinite “length of space” and the total charge is

$$Q = \int dy j^0(y). \quad (27)$$

As this is a constant, we may disregard it.

### 2.2. Form of Hamiltonian

Thus we obtain the form found (for the conventional theory) years ago by Lowell Brown [7]:

$$H = \frac{1}{4} \int dy dz j^0(y) |y - z| j^0(z) - \int dx \left\{ \frac{i}{2} \psi \gamma^0 \gamma^1 \partial_1 \psi - \frac{m}{2} \psi \gamma^0 \psi \right\}. \quad (28)$$

This resembles  $\phi^4$  theory, and for the same reason, we cannot calculate the  $\mathcal{C}$  operator perturbatively.

### 2.3. Functional integral

It may be useful to rederive the Hamiltonian using the functional integral approach. The partition function is

$$Z = \int [d\psi][d\phi] e^{i \int dt dx L}, \quad (29)$$

where in the Coulomb gauge (with  $A^0 = \phi$ )

$$L = \frac{1}{2}(\partial_1 \phi)^2 - i j^0 \phi - \frac{1}{2} \psi \gamma^0 \gamma^\mu \frac{1}{i} \partial_\mu \psi - \frac{m}{2} \psi \gamma^0 \psi. \quad (30)$$

We integrate out the scalar potential  $\phi$  by completing the square,

$$\frac{1}{2}(\partial_1 \phi)^2 - i j^0 \phi = \frac{1}{2} \left( \partial_1 \phi + \frac{i}{\partial_1^2} j^0 \right)^2 - \frac{1}{2} j^0 \frac{1}{\partial_1^2} j^0, \quad (31)$$

where

$$\frac{1}{\partial_1^2} = \frac{1}{2} |x - y|. \quad (32)$$

The functional integral on  $\phi$  is carried out over real values of  $\phi(x)$ . Then, the partition function can be written as

$$Z = \int [d\psi] e^{i \int dt dx (\pi \dot{\psi} - \mathcal{H})}, \quad (33)$$

where the momentum conjugate to  $\psi$  is

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = \frac{i}{2} \psi. \quad (34)$$

The result for  $H$ , given in (28), is reproduced. Because the sign of the quartic term in  $H$  is reversed, presumably we can no longer regard  $[d\psi]$  in (33) as over “real” values of  $\psi$ . Since Grassmann integration is a formal procedure, it is not immediately clear how to proceed. Henceforth, we will set the fermion mass  $m = 0$ , so we will refer to the Schwinger model proper.

### 2.4. ETCR of currents

It is easy to check that

$$[j^0(x, t), j^0(y, t)] = 0. \quad (35)$$

However, it requires a point-splitting calculation [which does not modify (35)] to verify that

$$[j^0(x, t), j^1(y, t)] = -\frac{ie^2}{\pi} \frac{\partial}{\partial x} \delta(x - y). \quad (36)$$

The key element in the latter is that the singular part of the 2-point fermion correlation function is given by the free Green’s function:

$$\langle \psi_\alpha(x) (\psi(y) \gamma^0)_\beta \rangle = \frac{1}{i} G_{\alpha\beta}(x - y), \quad (37a)$$

$$G(z) = -\frac{1}{2\pi} \frac{\gamma_\mu z^\mu}{z^2 + i\epsilon}. \quad (37b)$$

This agrees with the massless fermion propagator found in (59b) below.

### 2.5. Conservation of electric charge

The electric current is exactly conserved:

$$\partial_0 j^0 = \frac{1}{i} [j^0, H] = -\partial_1 j^1, \quad (38a)$$

or

$$\partial_\mu j^\mu = 0. \quad (38b)$$

### 2.6. Axial-vector anomaly

In 2-dimensions, the dual current is

$$*j^\mu = \epsilon^{\mu\nu} j_\nu, \quad *j^0 = j_1, \quad *j^1 = j^0. \quad (39)$$

Now, using the above commutator between  $j^0$  and  $j^1$ , we find

$$\begin{aligned} \partial_0 *j^0 &= \partial_0 j_1 = \frac{1}{i} [j_1, H] \\ &= -\partial_1 j^0 + \frac{1}{i} \left[ j_1(x), \frac{1}{4} \int dy dz j^0(y) |y - z| j^0(z) \right], \end{aligned} \quad (40)$$

so from (36) this can be rewritten as

$$\begin{aligned} \partial_\mu *j^\mu(x) &= -\frac{e^2}{2\pi} \int dy dz \partial_x \delta(x - y) |y - z| j^0(z) \\ &= -\frac{ie^2}{\pi} \partial_x A^0 = \frac{ie^2}{\pi} E. \end{aligned} \quad (41)$$

This is the two-dimensional version of the famous Schwinger-Adler-Bell-Jackiw anomaly [9].

### 2.7. Schwinger mass generation

Combine the current conservation and axial-current non-conservation:

$$\partial_1 [\partial_0 j^0 + \partial_1 j^1 = 0] \quad (42a)$$

$$\partial_0 [\partial_0 j^1 + \partial_1 j^0 = \frac{ie^2}{\pi} E], \quad (42b)$$

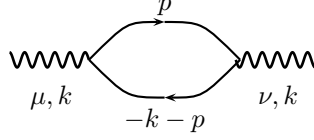
together with the Maxwell equation (22) to obtain  $(\partial^2 = -\partial_0^2 + \partial_1^2)$

$$\left( \partial^2 + \frac{e^2}{\pi} \right) j^1 = 0. \quad (43)$$

This corresponds to a *spacelike singularity*, a pole at

$$k^2 = -\partial^2 = \frac{e^2}{\pi}, \quad (44)$$

implying complex energies!



**Figure 1.** Lowest-order vacuum polarization graph

### 2.8. Perturbation theory

This result is consistent with perturbation theory, where in general we expect all we have to do is replace

$$e \rightarrow ie. \quad (45)$$

In fact, the Schwinger mass comes from one-loop vacuum polarization. In particular, the  $\mathcal{C}$  operator appears to have no effect on the weak-coupling expansion [10]. In general, it appears only ephemerally in the functional integral formulation [11].

Let us sketch the Feynman diagrammatic argument for the mass generation mechanism in the modified Schwinger model. That is, we calculate the vacuum polarization operator for massless QED in lowest order as shown in figure 1. This corresponds to

$$\Pi^{\mu\nu} = -(ie)^2 \text{tr} \int \frac{(d^d p)}{(2\pi)^d} \gamma^\mu \frac{-\gamma p}{p^2} \gamma^\nu \frac{\gamma(k-p)}{m^2 + (p-k)^2}, \quad (46)$$

where we have regulated the integral by working in  $d$  dimensions. The trace is evaluated as

$$\text{tr} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = d \left( g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} \right). \quad (47)$$

The denominators are combined according to

$$\frac{1}{p^2(p-k)^2} = \int_0^\infty ds s \int_0^1 du e^{-is\chi}, \quad (48)$$

with

$$\chi = (1-u)p^2 + u(p-k)^2 = (p-ku)^2 + k^2 u(1-u). \quad (49)$$

Now the integration variable is shifted,  $p \rightarrow p + ku$ , and odd terms disappear upon symmetric integration, leaving us with

$$\begin{aligned} \Pi^{\mu\nu} = & -(ie)^2 d \int_0^\infty ds s \int_0^\infty du \int \frac{d^d p}{(2\pi)^d} \left[ 2p^\mu p^\nu - 2k^\mu k^\nu u(1-u) \right. \\ & \left. - g^{\mu\nu} (p^2 - k^2 u(1-u)) \right] e^{-sp^2} e^{-su(1-u)k^2}. \end{aligned} \quad (50)$$

Note that in  $d = 2$  dimensions, the contraction (trace) of the tensor,  $\Pi^\mu{}_\mu$ , vanishes, which indicates that, apparently, the only gauge-invariant result could be zero. That this is incorrect is the result of the quantum anomaly, which appears only by setting

$d = 2$  at the end of the calculation. Proceeding onward, we use the following momentum integrals:

$$\int d^d p e^{-sp^2} = \left( \int dp e^{-sp^2} \right)^d = \left( \frac{\pi}{s} \right)^{d/2}, \quad (51a)$$

$$\int d^d p p^\mu p^\nu e^{-sp^2} = -g^{\mu\nu} \frac{1}{d} \frac{d}{ds} \int d^d p e^{-sp^2} = \frac{1}{2s} \left( \frac{\pi}{s} \right)^{d/2} g^{\mu\nu}. \quad (51b)$$

When the  $s$  integrals are now carried out, we obtain an explicitly gauge-invariant form:

$$\Pi^{\mu\nu} = -2^{1-d} (ie)^2 \pi^{-d/2} d\Gamma(2-d/2) \int_0^1 du [k^2 u(1-u)]^{d/2-1} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right), \quad (52)$$

which is divergent for  $d = 4$ , but finite for  $d = 2$ :

$$\Pi^{\mu\nu} = -\frac{(ie)^2}{\pi} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right). \quad (53)$$

We can similarly calculate the correction to the fermion propagator, given by the mass operator, using the covariant photon propagator,

$$\Sigma = (ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu \gamma(p-k) \gamma^\nu}{k^2 (p-k)^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (54)$$

The gamma matrix structures are reduced as

$$\gamma^\lambda \gamma(p-k) \gamma_\lambda = (d-2) \gamma(p-k), \quad (55a)$$

$$\gamma k \gamma(p-k) \gamma k = k^2 \gamma(p-k) + \gamma k [(p-k)^2 + k^2 - p^2], \quad (55b)$$

and so the integral, after the substitution  $k \rightarrow k + pu$ , can be evaluated as

$$\begin{aligned} & \gamma p \int_0^\infty ds s \int_0^1 du \left( \frac{\pi}{s} \right)^{d/2} \left[ (d-3)(1-u) - u - p^2 \frac{d}{dp^2} \right] e^{-su(1-u)p^2} \\ &= \gamma p \int_0^1 du \left( \frac{d}{2} - 1 \right) (1-2u) \pi^{d/2} \Gamma(2-d/2) [p^2 u(1-u)]^{d/2-2} \\ &= \gamma p \pi^{d/2} \Gamma(2-d/2) (p^2)^{d/2-2} \left( \frac{\Gamma(d/2) \sqrt{2\pi}}{\Gamma(d/2-1/2) 2^{d-5/2}} - 2 \frac{\Gamma(d/2)^2}{\Gamma(d-1)} \right) \rightarrow 0, \end{aligned} \quad (56)$$

as  $d \rightarrow 2$ , that is, the mass operator vanishes.

In the conventional theory, the vacuum polarization, iterated, yields the boson mass generated in the Schwinger model:

$$D(k) = \frac{1}{k^2} - \frac{1}{k^2} \frac{e^2}{\pi} \frac{1}{k^2} + \dots = \frac{1}{k^2 + e^2/\pi}, \quad (57)$$

while there is no correction to the fermion propagator,

$$iG(p) = \frac{1}{\gamma p}. \quad (58)$$

The corresponding iterated one-loop propagators in coordinate space are

$$D(x) = \frac{1}{2\pi} K_0(mx), \quad (59a)$$

$$G(x) = -\frac{1}{2\pi} \frac{\gamma^\mu x_\mu}{x^2}, \quad (59b)$$



where in the conventional theory  $m^2 = e^2/\pi$ , while the mass-squared is reversed in sign in the  $\mathcal{PT}$  theory. These propagators are given in terms of the Euclidean distance  $x = \sqrt{x^2}$ . Of course, in other gauges, there are corrections to the fermion propagator. However, in Schwinger’s words [6], “the detailed physical interpretation of the Green’s function is rather special and apart from our main purpose.”

### 3. Conclusions

Perturbation theory evidently fails to give a positive spectrum to the massless  $\mathcal{PT}$ -symmetric electrodynamics in 2 dimensions. More generally, this reflects the lack of unitarity of the  $S$ -matrix for the theory. This phenomenon has already been noticed by other authors: For example, Jones [12] shows for localized non-Hermitian potentials probability is not conserved, unless the Hilbert space metric is changed; and Smilga [13] shows that the  $S$ -matrix of a non-Hermitian theory defined in terms of “transition amplitudes between *conventional* asymptotic states is not unitary.”

Therefore, it seems that nonperturbative effects (strong field effects) presumably resolve this issue. It is necessary to do more than merely compute the  $\mathcal{Q}$  operator in field theory, which as we have seen makes no explicit appearance in the functional or perturbative formulation of the theory, but we must determine the asymptotic states, presumably defined dynamically. This is a formidable problem.

Previous work on these questions has concentrated on quantum-mechanical and scalar-field theory examples. Clearly there are issues unsolved relating to fermions and gauge theories in the  $\mathcal{PT}$ -context. In particular, the formal technique of Grassmann integration needs to be redeveloped. One must find ways to reformulate usual field theoretic tricks, such as bosonization.

Another illustration of the failure of perturbation theory to capture the essence of a  $\mathcal{PT}$  theory is discussed in the Appendix.

### Appendix A. Zero-Dimensions

We contrast the zero-dimensional partition functions for a conventional and a  $\mathcal{PT}$ -symmetric  $x^{2+N}$  theory:

$$Z_N^c(K) = \int_{-\infty}^{\infty} dx e^{-x^2 - gx^{2+N} - Kx}, \quad (1.1a)$$

$$Z_N(K) = \int_C dx e^{-x^2 - gx^2(ix)^N - Kx}. \quad (1.1b)$$

Such examples were discussed earlier in [14]. The contour  $C$  in the latter integral is taken in the lower half plane, along Stokes wedges centered on the lines

$$\theta = -\frac{N\pi}{4+2N}, \quad \theta = \pi - \frac{N\pi}{4+2N}, \quad (1.2)$$

which have width  $2\pi/(4+2N)$ , so that the integrand decays exponentially fast.

Note that the  $\mathcal{PT}$ -symmetric theory has a perturbation series that doesn't appear to know about the path of integration:

$$\begin{aligned} Z_N(K) &= \sqrt{\pi} \exp \left[ g \left( -i \frac{d}{dK} \right)^{N+2} \right] e^{K^2/4} \\ &= \sqrt{\pi} e^{K^2/8} \sum_{n=0}^{\infty} \left( \frac{(-1)^N g}{2^{1+N/2}} \right)^n \frac{1}{n!} D_{n(N+2)} \left( \frac{iK}{\sqrt{2}} \right), \end{aligned} \quad (1.3)$$

where  $D_m(x)$  is the parabolic cylinder function.

When we set  $N = 2$  and  $K = 0$  we get the  $-x^4$  theory without sources, for which we have the closed form for the vacuum amplitude

$$Z_2(0) = \frac{\pi}{4\sqrt{g}} e^{-1/8g} \left[ I_{1/4} \left( \frac{1}{8g} \right) + I_{-1/4} \left( \frac{1}{8g} \right) \right]. \quad (1.4)$$

The conventional theory in the same situation has the closed form

$$Z_2^c(0) = \frac{1}{2\sqrt{g}} e^{1/8g} K_{1/4} \left( \frac{1}{8g} \right). \quad (1.5)$$

Directly from (1.4), or from the previous expansion (1.3), we find the weak-coupling expansion ( $g \rightarrow 0$ )

$$Z_2(0) \sim \sqrt{\pi} \left( 1 + \frac{3}{4}g + \frac{105}{32}g^2 + \dots \right); \quad (1.6)$$

the expansion of  $Z_2^c$  differs only in the sign of  $g$ . The conventional theory is Borel summable, while the  $\mathcal{PT}$  series is not.

However, the correspondence is not so simple in strong-coupling. Even the leading prefactors in the strong-coupling expansions are different: ( $g \rightarrow \infty$ )

$$Z_2^c(0) \sim \frac{\sqrt{2}\pi}{2g^{1/4}\Gamma(3/4)} \left[ 1 - \frac{1}{4\sqrt{g}} \frac{\Gamma(3/4)}{\Gamma(5/4)} + \dots \right], \quad (1.7a)$$

$$Z_2(0) \sim \frac{\pi}{2g^{1/4}\Gamma(3/4)} \left[ 1 + \frac{1}{4\sqrt{g}} \frac{\Gamma(3/4)}{\Gamma(5/4)} + \dots \right]. \quad (1.7b)$$

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